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Computing Elliptic Integrals by Duplication

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Summary. Logarithms, arctangents, and elliptic integrals of all three kinds (including complete integrals) are evaluated numerically by successive applications of the duplication theorem. When the convergence is improved by including a fixed number of terms of Taylor's series, the error ultimately decreases by a factor of 4096 in each cycle of iteration. Except for Cauchy principal values there is no separation of cases according to the values of the variables, and no serious cancellations occur if the variables are real and nonnegative. Only rational operations and square roots are required. An appendix contains a recurrence relation and two new representations (in terms of elementary symmetric functions and power sums) for R-polynomials, as well as an upper bound for the error made in truncating the Taylor series of an R-function.

Subject Classifications. AMS(MOS): 65D20; CR: 5.12.

1. Introduction

Incomplete elliptic integrals are usually computed by successive Landen or Gauss transformations or by infinite series. Both methods are reviewed by Van de Vel [15] and Gautschi [10, pp. 19–21, 51–67], who cite many papers including the work of Bulirsch on Bartky transformations for integrals of the third kind and the work of Luke on Padé approximations. Additional references, some of which have appeared since Gautschi's review, are [9, 8, 7, 12], and [6]. In the first of these Franke proposes a series calculation of the third integral in which convergence is achieved or improved by one preliminary application of the duplication theorem. In [4, pp. 343–344] Carlson gives an algorithm for computing the first integral by successive applications of the duplication theorem, which decreases the differences between the variables by a factor of four in each application. The resulting rate of convergence is not competitive with other methods. The present paper proposes computation of all three integrals by a modification of this procedure: after the differences between

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the variables have been made small by successive duplications, the integrals are expanded in multiple Taylor series and terms up to fifth order are kept. The resulting algorithms require only rational operations and square roots. They converge fast enough, even though linearly, to compete with methods having quadratic convergence, unless abnormally high precision is required. Except for Cauchy principal values all cases (including complete, degenerate, circular, and hyperbolic cases) are computed by the same procedure with no need for special precautions to avoid loss of significant figures through cancellation. In this respect the method is superior to Gauss, Bartky, and Padé methods for the third integral. The precision may be chosen at will, and explicit error bounds are given. The algorithms are expected to be usable for complex values of the variables but have been tested and are stated only for real values.

Fortran implementations of the algorithms can be obtained from the author on request and will be submitted for publication elsewhere.

As a standard form for the integral of the first kind (see [6, § 9.2]), we choose

$$R_F(x,y,z)=\tfrac{1}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-\frac{1}{2}}\,dt, \tag{1.1}$$

where the variables x, y, z are nonnegative and at most one of them is 0. This function is symmetric and homogeneous of degree $-\frac{1}{2}$ in x, y, z and is normalized so that $R_F(x,x,x)=x^{-\frac{1}{2}}$. When two of the variables are equal, R_F degenerates to an elementary function,

$$R_C(x,y)=R_F(x,y,y)=\tfrac{1}{2}\int_0^\infty (t+x)^{-\frac{1}{2}}(t+y)^{-1}\,dt. \tag{1.2}$$

This is a logarithm if $0<y<x$ and an inverse circular function if $0\leqq x<y$ (see § 4). If $y<0$ we define R_C to be half the Cauchy principal value of the integral.

As a standard integral of the third kind (see [16]), we choose

$$R_J(x,y,z,\rho)=\tfrac{3}{2}\int_0^\infty [(t+x)(t+y)(t+z)]^{-\frac{1}{2}}(t+\rho)^{-1}\,dt, \tag{1.3}$$

where $\rho \neq 0$. This function is symmetric in x, y, z, homogeneous of degree $-\frac{3}{2}$ in x, y, z, ρ, and normalized so that $R_J(x,x,x,x)=x^{-\frac{3}{2}}$. *If* $\rho<0$ the Cauchy principal value is taken. If ρ equals one of the other variables, R_J degenerates to an integral of the second kind,

$$R_D(x,y,z)=R_J(x,y,z,z)=\tfrac{3}{2}\int_0^\infty (t+x)^{-\frac{1}{2}}(t+y)^{-\frac{1}{2}}(t+z)^{-\frac{3}{2}}\,dt. \tag{1.4}$$

The relations of these integrals to other standard forms are given in § 4. If $x=0$ the integrals are said to be complete.

2. Statement of Algorithms

The following algorithms for computing R_F, R_C, R_J, and R_D are proved in § 5. The algorithm for R_C is used repeatedly in the algorithm for R_J.

Algorithm 1. Let $x_0 \geqq 0$, $y_0 > 0$, and $z_0 > 0$. For $n = 0, 1, 2, \ldots$, let

$$\lambda_n = (x_n y_n)^{\frac{1}{2}} + (x_n z_n)^{\frac{1}{2}} + (y_n z_n)^{\frac{1}{2}}, \qquad \mu_n = (x_n + y_n + z_n)/3, \tag{2.1}$$

$$x_{n+1} = (x_n + \lambda_n)/4, \qquad y_{n+1} = (y_n + \lambda_n)/4, \qquad z_{n+1} = (z_n + \lambda_n)/4, \tag{2.2}$$

$$X_n = 1 - (x_n/\mu_n), \qquad Y_n = 1 - (y_n/\mu_n), \qquad Z_n = 1 - (z_n/\mu_n), \tag{2.3}$$

$$\varepsilon_n = \max\{|X_n|, |Y_n|, |Z_n|\}, \qquad s_n^{(m)} = (X_n^m + Y_n^m + Z_n^m)/2m, \quad (m = 2, 3). \tag{2.4}$$

Then $\varepsilon_n = O(4^{-n})$ as $n \to \infty$. If $\varepsilon_n < 1$ then

$$R_F(x_0, y_0, z_0) = \mu_n^{-\frac{1}{2}}\left[1 + \tfrac{1}{5} s_n^{(2)} + \tfrac{1}{7} s_n^{(3)} + \tfrac{1}{6}(s_n^{(2)})^2 + \tfrac{3}{11} s_n^{(2)} s_n^{(3)} + r_n\right], \tag{2.5}$$

$$|r_n| < \frac{\varepsilon_n^6}{4(1-\varepsilon_n)} \quad \text{and} \quad r_n \sim \tfrac{5}{26}(s_n^{(2)})^3 + \tfrac{3}{26}(s_n^{(3)})^2, \qquad n \to \infty. \tag{2.6}$$

Although this algorithm has linear rather than quadratic convergence, it is quite fast because r_n is of order $4^{-6n} = (4096)^{-n}$. The number of cycles required for given accuracy increases slowly with increasing ratio of the largest of x_0, y_0, z_0 to the next largest (see § 3). Computation of λ_n can be accomplished with two square roots if desired, since

$$(x_n y_n)^{\frac{1}{2}} = x_n y_n z_n / (x_n z_n)^{\frac{1}{2}} (y_n z_n)^{\frac{1}{2}}, \qquad x_n \neq 0. \tag{2.7}$$

Note that $Z_n = -X_n - Y_n$. It may be advantageous to rewrite (2.5) in terms of $E_2 = X_n Y_n + X_n Z_n + Y_n Z_n = X_n Y_n - Z_n^2$ and $E_3 = X_n Y_n Z_n$ by using (5.10). If $y_0 = z_0$ Algorithm 1 reduces to the following algorithm for R_C.

Algorithm 2. Let $x_0 \geqq 0$ and $y_0 > 0$. For $n = 0, 1, 2, \ldots$, let

$$\lambda_n = 2(x_n y_n)^{\frac{1}{2}} + y_n, \qquad x_{n+1} = (x_n + \lambda_n)/4, \qquad y_{n+1} = (y_n + \lambda_n)/4, \tag{2.8}$$

$$\mu_n = (x_n + 2y_n)/3, \qquad s_n = (y_n - x_n)/3\mu_n. \tag{2.9}$$

Then $s_n = O(4^{-n})$ as $n \to \infty$. If $|s_n| < \frac{1}{2}$ then

$$R_C(x_0, y_0) = \mu_n^{-\frac{1}{2}}\left(1 + \tfrac{3}{10} s_n^2 + \tfrac{1}{7} s_n^3 + \tfrac{3}{8} s_n^4 + \tfrac{9}{22} s_n^5 + r_n\right), \tag{2.10}$$

$$|r_n| < \frac{16|s_n|^6}{1 - 2|s_n|} \quad \text{and} \quad r_n \sim \tfrac{159}{208} s_n^6, \qquad n \to \infty. \tag{2.11}$$

To get an alternative algorithm with simpler coefficients but slower convergence, take (2.8) together with

$$t_n = 1 - (y_n/x_n). \tag{2.9a}$$

Then $t_n = O(4^{-n})$ as $n \to \infty$. If $|t_n| < 1$ then

$$R_C(x_0, y_0) = x_n^{-\frac{1}{2}}\left(\sum_{m=0}^{5} \frac{1}{2m+1} t_n^m + r_n\right), \tag{2.10a}$$

$$|r_n| < \frac{1}{13} \frac{|t_n|^6}{1 - |t_n|} \quad \text{and} \quad r_n \sim \frac{1}{13} t_n^6, \qquad n \to \infty. \tag{2.11a}$$

To compute the Cauchy principal value of $R_C(x,y)$ if $y<0$, Algorithm 2 may be used after first applying the transformation

$$R_C(x,y)=\left(\frac{x}{x-y}\right)^{\frac{1}{2}} R_C(x-y,\,-y), \qquad x\geqq 0,\ y<0. \tag{2.12}$$

The Cauchy principal value is 0 if $x=0$ and strictly positive if $x>0$.

Algorithm 3. Let $x_0\geqq 0$, $y_0>0$, $z_0>0$, and $\rho_0>0$. For $n=0,1,2,\ldots$, let

$$\lambda_n=(x_n y_n)^{\frac{1}{2}}+(x_n z_n)^{\frac{1}{2}}+(y_n z_n)^{\frac{1}{2}}, \qquad \mu_n=(x_n+y_n+z_n+2\rho_n)/5, \tag{2.13}$$

$$\begin{aligned} &x_{n+1}=(x_n+\lambda_n)/4, \quad y_{n+1}=(y_n+\lambda_n)/4, \quad z_{n+1}=(z_n+\lambda_n)/4,\\ &\rho_{n+1}=(\rho_n+\lambda_n)/4, \end{aligned} \tag{2.14}$$

$$\begin{aligned} &X_n=1-(x_n/\mu_n), \quad Y_n=1-(y_n/\mu_n), \quad Z_n=1-(z_n/\mu_n),\\ &P_n=1-(\rho_n/\mu_n), \end{aligned} \tag{2.15}$$

$$\varepsilon_n=\max\{|X_n|,|Y_n|,|Z_n|,|P_n|\}, \tag{2.16}$$

$$s_n^{(m)}=(X_n^m+Y_n^m+Z_n^m+2P_n^m)/2m, \qquad (m=2,3,4,5), \tag{2.17}$$

$$\alpha_n=[\rho_n(x_n^{\frac{1}{2}}+y_n^{\frac{1}{2}}+z_n^{\frac{1}{2}})+(x_n y_n z_n)^{\frac{1}{2}}]^2, \qquad \beta_n=\rho_n(\rho_n+\lambda_n)^2. \tag{2.18}$$

Then $\varepsilon_n=O(4^{-n})$ as $n\to\infty$. If $\varepsilon_n<1$ then

$$\begin{aligned} R_J(x_0,y_0,z_0,\rho_0)=3\sum_{m=0}^{n-1} 4^{-m} R_C(\alpha_m,\beta_m) &\\ +4^{-n}\mu_n^{-\frac{3}{2}}[1+\tfrac{3}{7}s_n^{(2)}+\tfrac{1}{3}s_n^{(3)}+\tfrac{3}{22}(s_n^{(2)})^2+\tfrac{3}{11}s_n^{(4)} &\\ +\tfrac{3}{13}s_n^{(2)}s_n^{(3)}+\tfrac{3}{13}s_n^{(5)}+r_n], & \end{aligned} \tag{2.19}$$

$$|r_n|<\frac{3\varepsilon_n^6}{(1-\varepsilon_n)^{\frac{3}{2}}}, \tag{2.20}$$

$$r_n\sim\tfrac{-1}{10}(s_n^{(2)})^3+\tfrac{3}{10}(s_n^{(3)})^2+\tfrac{3}{5}s_n^{(2)}s_n^{(4)}, \qquad n\to\infty. \tag{2.21}$$

Note that $s_n^{(m)}$ needs to be calculated for only a single value of n. A little algebra shows that $\alpha_n-\beta_n=(x_n-\rho_n)(y_n-\rho_n)(z_n-\rho_n)$, but (2.18) avoids cancellation in finding α_n when ρ_n is large. Algorithm 2 is used to calculate $R_C(\alpha_m,\beta_m)$ in (2.19). Since $\alpha_m-\beta_m=O(4^{-3m})$ the number of cycles needed in Algorithm 2 decreases rapidly as m increases. In addition to the square roots needed in Algorithm 2, the square roots of x_n, y_n, and z_n are used in each cycle to compute λ_n and α_n. Actually the square roots of $x_n z_n$ and $y_n z_n$ suffice to calculate both λ_n and α_n (see (2.7)), but avoiding a third square root may not be worth the extra multiplications and divisions.

To compute the Cauchy principal value of $R_J(x,y,z,\rho)$ if $\rho<0$, the preceding algorithms may be used after first applying the transformation [16, (4.7)]

$$\begin{aligned} (y-\rho)R_J(x,y,z,\rho)=(\gamma-y)R_J(x,y,z,\gamma)-3R_F(x,y,z) &\\ +3R_C(xz/y,\rho\gamma/y), \qquad \gamma=y+\frac{(z-y)(y-x)}{y-\rho}. & \end{aligned} \tag{2.22}$$

If $\rho<0$ and the other variables are labeled so that $0\leqq x\leqq y\leqq z$, then $y\leqq\gamma\leqq z$. (The transformation is not limited to this case; see [16, Table 1].)

Given strictly positive x, y, z there exists $\rho<0$ such that the Cauchy principal value of $R_J(x,y,z,\rho)$ vanishes (see §5). Near the zero of R_J there will be cancellation between the terms on the right side of (2.22), leading to loss of significant figures.

If $\rho_0=z_0$ Algorithm 3 reduces to the following algorithm for R_D.

Algorithm 4. Let $x_0\geqq 0$, $y_0>0$, and $z_0>0$. For $n=0,1,2,\ldots$, let

$$\lambda_n=(x_n y_n)^{\frac{1}{2}}+(x_n z_n)^{\frac{1}{2}}+(y_n z_n)^{\frac{1}{2}}, \qquad \mu_n=(x_n+y_n+3z_n)/5, \tag{2.23}$$

$$x_{n+1}=(x_n+\lambda_n)/4, \qquad y_{n+1}=(y_n+\lambda_n)/4, \qquad z_{n+1}=(z_n+\lambda_n)/4, \tag{2.24}$$

$$X_n=1-(x_n/\mu_n), \qquad Y_n=1-(y_n/\mu_n), \qquad Z_n=1-(z_n/\mu_n), \tag{2.25}$$

$$\varepsilon_n=\max\{|X_n|,|Y_n|,|Z_n|\}, \qquad s_n^{(m)}=(X_n^m+Y_n^m+3Z_n^m)/2m, \quad (m=2,3,4,5). \tag{2.26}$$

Then $\varepsilon_n=O(4^{-n})$ as $n\to\infty$. If $\varepsilon_n<1$ then

$$\begin{aligned} R_D(x_0,y_0,z_0)=3\sum_{m=0}^{n-1}\frac{4^{-m}}{z_m^{\frac{1}{2}}(z_m+\lambda_m)} &+4^{-n}\mu_n^{-\frac{3}{2}}\big[1+\tfrac{3}{7}s_n^{(2)}+\tfrac{1}{3}s_n^{(3)}+\tfrac{3}{22}(s_n^{(2)})^2+\tfrac{3}{11}s_n^{(4)}\\ &+\tfrac{3}{13}s_n^{(2)}s_n^{(3)}+\tfrac{3}{13}s_n^{(5)}+r_n\big], \end{aligned} \tag{2.27}$$

$$|r_n|<\frac{3\varepsilon_n^6}{(1-\varepsilon_n)^{\frac{3}{2}}}, \tag{2.28}$$

$$r_n\sim\tfrac{-1}{10}(s_n^{(2)})^3+\tfrac{3}{10}(s_n^{(3)})^2+\tfrac{3}{5}s_n^{(2)}s_n^{(4)}, \qquad n\to\infty. \tag{2.29}$$

3. Numerical Examples

To illustrate the use of Algorithms 1 and 4 we compute the lemniscate constants (see Todd [14]),

$$\begin{aligned} A&=\int_0^1(1-s^4)^{-\frac{1}{2}}\,ds=R_F(0,1,2)=R_F(0,2,1),\\ B&=\int_0^1 s^2(1-s^4)^{-\frac{1}{2}}\,ds=\tfrac{1}{3}R_D(0,2,1). \end{aligned} \tag{3.1}$$

Reduction to the standard forms (1.1) and (1.4) is accomplished by substituting $s=t^{\frac{1}{2}}(t+2)^{-\frac{1}{4}}$ in the first integral and $s=(t+1)^{-\frac{1}{4}}$ in the second. Letting $x_0=0$, $y_0=2$, and $z_0=1$ in Algorithms 1 and 4, we find $z_n=\frac{1}{2}(x_n+y_n)=\mu_n$, $Z_n=0$, $Y_n=-X_n$, $\varepsilon_n=X_n$, $s_n^{(2)}=\frac{1}{2}X_n^2$, $s_n^{(3)}=s_n^{(5)}=0$, and $s_n^{(4)}=(s_n^{(2)})^2$. A programmable hand calculator showing ten significant figures gave the following values:

n	x_n	$z_n=\mu_n$	$X_n=\varepsilon_n$	λ_n
0	0.0000000000	1.0000000000	1.0000000000	1.414213562
1	0.3535533905	0.6035533905	0.4142135625	1.729031809
2	0.5206463000	0.5831463000	0.1071772212	1.744399246
3	0.5662613865	0.5818863865	0.0268523210	

Substitution in (2.5) and (2.6) yielded

$$A=R_F(0,2,1)=(1.310934224)[1+0.0000721105+0.0000000022+r_3],$$
$$|r_3|<1.0\times 10^{-10}, \qquad A=1.3110288778.$$

Values obtained from (2.27) and (2.28) were

$$B=\tfrac{1}{3}R_D(0,2,1)=(0.4142135624+0.1379572872+0.0351635786)$$
$$+\frac{2.2529040 97}{3\times 64}[1+0.0001545 10+0.0000000053+r_3],$$
$$|r_3|<1.2\times 10^{-9}, \qquad B=0.5873344282+0.0117356891=0.5990701173.$$

Because of roundoff error the value of A is larger than the correct value by one unit in the last place, and the value of B is smaller by one unit. We have not used the value of π, which can now be calculated from the relation $\pi=4AB$.

We have kept terms of fifth order in the Taylor series, but higher terms can be generated by the recurrence relations (5.11) and (5.12). The recurrence should be numerically stable because the desired solution is dominant according to Theorem 1 in the Appendix. For very precise computations one might choose to keep terms of order n, where n is the number of duplications; then r_n would be of order $(4^{-n})^n$. However, quadratic convergence is better for very precise computations (cf. Brent [1]), and the algorithms in the present paper are intended primarily for precision up to 20S.

The convergence becomes slower with increasing ratio of the largest to the next largest of x_0, y_0, z_0 (in Legendre's notation, with increasing value of $k\,sin\,\varphi=\sin\theta\sin\varphi$). For example, in computing $R_F(x_0,y_0,z_0)$ with $2x_0=y_0=10^{-10}$ and $z_0=1$ (corresponding approximately to $\theta=\varphi=89.9996°$), we find $\varepsilon_n>1$ for $n\leqq 3$. To insure that $|r_n|<10^{-10}$ we must take $n=6$ in Algorithm 1:

$$x_6=y_6=6.4603496690\times 10^{-3}, \qquad z_6=6.7044903315\times 10^{-3},$$
$$X_6=Y_6=1.2440166\times 10^{-2}, \qquad \varepsilon_6=-Z_6=2.4880332\times 10^{-2},$$

$$R_F(5\times 10^{-11}, 10^{-10}, 1)$$
$$=(12.36384909)(1+0.0000464 27-0.0000000275+0.0000000009)$$
$$=12.36441982.$$

Algorithm 4 with $n=6$ yields

$$R_D(5\times 10^{-11}, 10^{-10}, 1)=3\times(11.21285262)+0.45470162=34.09325948.$$

In each case the value is smaller by one unit in the last place than a value obtained from ascending Landen transformations [3].

Algorithm 2 in conjunction with (4.9) to (4.13) is related to the algorithms for elementary functions given in [5]. Both methods start with successive duplications, but the convergence is improved in [5] by extrapolation and here by Taylor series. For comparison with the second example in [5, § 3] we compute $\pi/4 = \arctan 1 = R_C(1,2)$ [see (4.12)]. To make $|r_n| < 2 \times 10^{-10}$, it suffices that $|s_n| < 0.015$, and this is achieved for $n=2$:

n	x_n	y_n	s_n	λ_n
0	1.0000000000	2.0000000000	0.2000000000	4.8284271247
1	1.4571067811	1.7071067811	0.0513207883	4.8614288811
2	1.5796338981	1.6421338981	0.0128497663	

$$R_C(1,2) = (0.7853590144)(1 + 0.0000495535 + 0.0000000303 + 0.0000000010 + r_2),$$

$$|r_2| < 8 \times 10^{-11}, \qquad \frac{\pi}{4} = R_C(1,2) = 0.7853981630.$$

Because of roundoff error this value of $\pi/4$ is smaller than the correct value by four units in the last place. Only three square roots were extracted (in computing λ_0, λ_1, and $\mu_2^{-\frac{1}{2}}$), compared with four in [5]. (Since the earlier calculation was done to higher precision, the rounded values tabulated in [5] are correct in the last place.)

The relative error is of order 2 raised to the power $-12n$ for Algorithm 2 and the power $-n^2$ for the algorithm in [5]. For highly precise computation, however, the power $-2n^2$ could be achieved by taking n terms of the Taylor series $\Sigma a_m s_n^m$ in (2.10). The recurrence relation (5.14) for a_m is numerically stable because a_m is a dominant solution.

Some check values for Algorithm 2 are $\pi = 2R_C(0,1) = 4R_C(1,2) = 6R_C(3,4)$, $\ln 2 = 2R_C(9,8) = 3R_C(25,16)$, and $\ln 10 = 18R_C(121,40)$.

To illustrate the use of Algorithm 3 we compute $R_J(2,3,4,5)$. To insure $|r_n| < 5 \times 10^{-10}$, we require $\varepsilon_n < 0.023$; this is satisfied for $n=3$:

$$\begin{aligned}
\varepsilon_3 = X_3 &= 9.5541245(-3), & s_3^{(2)} &= 4.7894504(-5),\\
Y_3 &= 4.2462776(-3), & s_3^{(3)} &= 7.1778850(-8),\\
Z_3 &= -1.0615690(-3), & s_3^{(4)} &= 1.4938024(-9),\\
P_3 &= -6.3694160(-3), & s_3^{(5)} &= 6.0020157(-12),
\end{aligned}$$

where $p(q)$ means $p \times 10^q$. Nine square roots were extracted in computing $x_n^{\frac{1}{2}}, y_n^{\frac{1}{2}}, z_n^{\frac{1}{2}}$ for $n = 0, 1, 2$. Since $\beta_n - \alpha_n \ll \alpha_n$ even for $n=0$, only one additional square root is needed for each value of R_C:

n	α_n	β_n	$R_C(\alpha_n, \beta_n)$
0	9.382153602 (2)	9.442153595 (2)	3.257808092 (−2)
1	5.134301351 (2)	5.135238852 (2)	4.412989468 (−2)
2	4.292019536 (2)	4.292034184 (2)	4.826903996 (−2)

$$\begin{aligned} R_J(2,3,4,5) &= 3\times 4.662736959\times 10^{-2} \\ &\quad + 4^{-3}(2.943754800)^{-\frac{3}{2}}[1+0.0000020526 \\ &\quad + 0.0000000024 + 0.0000000001] \\ &= 0.1398821088 + 0.0030936879 = 0.1429757967. \end{aligned}$$

The last number inside the brackets includes both terms of fourth order in (2.19). The terms of fifth order are negligible and $|r_3| < 3\times 10^{-12}$. Thirteen square roots were extracted in all. A check on the value of R_J (which should be rounded to at most 8 S) is provided by (4.3), (4.1), and the tables in [8]:

$$\begin{aligned} R_J(2,3,4,5) &= (3/\sqrt{2})[F(45°, 1/\sqrt{2}) - \Pi(45°, 1/\sqrt{2}, 1/2)] \\ &= (3/\sqrt{2})(0.8260178763 - 0.7586184393) \\ &= 0.142975797. \end{aligned}$$

4. Other Standard Integrals

Legendre's standard integrals can be expressed as follows in terms of R_F, R_D, and R_J:

$$F(\varphi, k) = (\sin\varphi)\, R_F(\cos^2\varphi,\ 1-k^2\sin^2\varphi,\ 1), \tag{4.1}$$

$$\begin{aligned} E(\varphi, k) &= (\sin\varphi)\, R_F(\cos^2\varphi,\ 1-k^2\sin^2\varphi,\ 1) \\ &\quad - \tfrac{1}{3}k^2(\sin\varphi)^3 R_D(\cos^2\varphi,\ 1-k^2\sin^2\varphi,\ 1), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \Pi(\varphi, k, n) &= \int_0^{\varphi} (1+n\sin^2\theta)^{-1}(1-k^2\sin^2\theta)^{-\frac{1}{2}}\, d\theta \\ &= (\sin\varphi)\, R_F(\cos^2\varphi,\ 1-k^2\sin^2\varphi,\ 1) \\ &\quad - \frac{n}{3}(\sin\varphi)^3 R_J(\cos^2\varphi,\ 1-k^2\sin^2\varphi,\ 1,\ 1+n\sin^2\varphi), \end{aligned} \tag{4.3}$$

$$\begin{aligned} D(\varphi, k) &= \int_0^{\varphi} \sin^2\theta\,(1-k^2\sin^2\theta)^{-\frac{1}{2}}\, d\theta \\ &= \tfrac{1}{3}(\sin\varphi)^3 R_D(\cos^2\varphi,\ 1-k^2\sin^2\varphi,\ 1), \end{aligned} \tag{4.4}$$

$$K(k) = R_F(0,\ 1-k^2,\ 1), \tag{4.5}$$

$$E(k) = R_F(0,\ 1-k^2,\ 1) - \tfrac{1}{3}k^2 R_D(0,\ 1-k^2,\ 1). \tag{4.6}$$

Heuman's lambda function [11] is a variant of Legendre's third integral:

$$
\begin{aligned}
\Lambda(\alpha,\beta,\varphi)=&\frac{\cos^2\alpha\sin\beta\cos\beta}{(1-\cos^2\alpha\sin^2\beta)^{\frac{1}{2}}}\Big[(\sin\varphi)R_F(\cos^2\varphi,1-\sin^2\alpha\sin^2\varphi,1)\\
&+\frac{\sin^2\alpha\sin^3\varphi}{3(1-\cos^2\alpha\sin^2\beta)}\\
&\cdot R_J\left(\cos^2\varphi,1-\sin^2\alpha\sin^2\varphi,1,1-\frac{\sin^2\alpha\sin^2\varphi}{1-\cos^2\alpha\sin^2\beta}\right)\Big],
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
\frac{\pi}{2}\Lambda_0(\alpha,\beta)=&\Lambda(\alpha,\beta,\pi/2)\\
=&(\sin\beta)[R_F(0,\cos^2\alpha,1)-\tfrac{1}{3}(\sin\alpha)^2R_D(0,\cos^2\alpha,1)]\\
&\cdot R_F(\cos^2\beta,1-\cos^2\alpha\sin^2\beta,1)\\
&-\tfrac{1}{3}\cos^2\alpha\sin^3\beta R_F(0,\cos^2\alpha,1)\\
&\cdot R_D(\cos^2\beta,1-\cos^2\alpha\sin^2\beta,1).
\end{aligned}
\tag{4.8}
$$

Logarithms, inverse circular functions, and inverse hyperbolic functions can be expressed in terms of R_C (see [6, pp. 163, 186]):

$$\ln x=(x-1)R_C\left[\left(\frac{1+x}{2}\right)^2,x\right],\qquad x>0: \tag{4.9}$$

$$
\begin{aligned}
\arcsin x&=xR_C(1-x^2,1), && -1\leqq x\leqq 1;\\
\operatorname{arcsinh} x&=xR_C(1+x^2,1), && -\infty<x<\infty;
\end{aligned}
\tag{4.10}
$$

$$
\begin{aligned}
\arccos x&=(1-x^2)^{\frac{1}{2}}R_C(x^2,1), && 0\leqq x\leqq 1;\\
\operatorname{arccosh} x&=(x^2-1)^{\frac{1}{2}}R_C(x^2,1), && x\geqq 1;
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\arctan x&=xR_C(1,1+x^2), && -\infty<x<\infty;\\
\operatorname{arctanh} x&=xR_C(1,1-x^2), && -1<x<1;
\end{aligned}
\tag{4.12}
$$

$$
\begin{aligned}
\operatorname{arccot} x&=R_C(x^2,x^2+1), && 0\leqq x<\infty;\\
\operatorname{arccoth} x&=R_C(x^2,x^2-1), && x>1.
\end{aligned}
\tag{4.13}
$$

If x is close to 1, computation of R_C gives the value of $(\ln x)/(x-1)$ by (4.9) without the loss of significant figures that occurs when $\ln x$ and $x-1$ are computed separately.

Although $R_D(x,y,z)$ is easy to compute by Algorithm 3, it is not an ideal choice for a standard function because it is not symmetric in x, y, z. The symmetric standard integral of the second kind [6, § 9.2] is given by

$$2R_G(x,y,z)=zR_F(x,y,z)-\tfrac{1}{3}(z-x)(z-y)R_D(x,y,z)+(xy/z)^{\frac{1}{2}}. \tag{4.14}$$

An alternative to R_J (see [16, (2.10)] and [6, § 9.2]) is

$$2R_H(x,y,z,\rho)=3R_F(x,y,z)-\rho R_J(x,y,z,\rho). \tag{4.15}$$

The variants of Legendre's integrals used by Bulirsch [2] are

$$el1(x, k_c) = x R_F(1, 1+k_c^2 x^2, 1+x^2), \tag{4.16}$$

$$\begin{aligned} el2(x, k_c, a, b) = {} & a x R_F(1, 1+k_c^2 x^2, 1+x^2) \\ & + \tfrac{1}{3}(b-a) x^3 R_D(1, 1+k_c^2 x^2, 1+x^2), \end{aligned} \tag{4.17}$$

$$\begin{aligned} el3(x, k_c, p) = {} & x R_F(1, 1+k_c^2 x^2, 1+x^2) \\ & + \tfrac{1}{3}(1-p) x^3 R_J(1, 1+k_c^2 x^2, 1+x^2, 1+px^2), \end{aligned} \tag{4.18}$$

$$cel(k_c, p, a, b) = a R_F(0, k_c^2, 1) + \tfrac{1}{3}(b-pa) R_J(0, k_c^2, 1, p). \tag{4.19}$$

5. Proof of the Algorithms

Algorithms 1 and 3 are the basic ones, the other two being special cases. The duplication theorems for R_F [6, (9.6−1)] and R_J [16, § 8] (in this reference R_C is denoted by R_f) imply that $R_F(x_n, y_n, z_n)$ is independent of n and

$$R_J(x_n, y_n, z_n, \rho_n) = 3 R_C(\alpha_n, \beta_n) + \tfrac{1}{4} R_J(x_{n+1}, y_{n+1}, z_{n+1}, \rho_{n+1}). \tag{5.1}$$

It follows that

$$R_F(x_0, y_0, z_0) = R_F(x_n, y_n, z_n), \tag{5.2}$$

$$R_J(x_0, y_0, z_0, \rho_0) = 3 \sum_{m=0}^{n-1} 4^{-m} R_C(\alpha_m, \beta_m) + 4^{-n} R_J(x_n, y_n, z_n, \rho_n). \tag{5.3}$$

Note that (2.2) implies $x_{n+1} - y_{n+1} = (x_n - y_n)/4$; similarly all differences between x_n, y_n, z_n, ρ_n are reduced by a factor of four when n increases by unity. When n is sufficiently large, we expand the last members of (5.2) and (5.3) in multiple Taylor series to obtain (2.5) and (2.19). The two cases can be treated together, since (1.1), (1.3), and [6, (6.8−6)] identify R_F and R_J as special cases of the R-function:

$$R_F(x_n, y_n, z_n) = R_{-\frac{1}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x_n, y_n, z_n), \tag{5.4}$$

$$\begin{aligned} R_J(x_n, y_n, z_n, \rho_n) &= R_{-\frac{3}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1; x_n, y_n, z_n, \rho_n) \\ &= R_{-\frac{3}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; x_n, y_n, z_n, \rho_n, \rho_n). \end{aligned} \tag{5.5}$$

The last equality follows from [6, Theorem 5.2−4].

The Taylor series are greatly simplified by expanding about the arithmetic mean μ_n of the variables (more precisely, by expanding in powers of relative deviations from the mean). Define $\mu_n, X_n, Y_n, \ldots$ by (2.1) and (2.3) for R_F and by (2.13) and (2.15) for R_J. Using homogeneity and (A.5) (see the Appendix), we find the Taylor series

$$R_F(x_n, y_n, z_n) = \mu_n^{-\frac{1}{2}} R_{-\frac{1}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1-X_n, 1-Y_n, 1-Z_n) = \mu_n^{-\frac{1}{2}} \sum_{m=0}^{\infty} v_m, \tag{5.6}$$

$$\begin{aligned} R_J(x_n, y_n, z_n, \rho_n) &= \mu_n^{-\frac{3}{2}} R_{-\frac{3}{2}}(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1-X_n, 1-Y_n, 1-Z_n, 1-P_n, 1-P_n) \\ &= \mu_n^{-\frac{3}{2}} \sum_{m=0}^{\infty} w_m, \end{aligned} \tag{5.7}$$

where the terms of degree m in $X_n, Y_n, \dots$ constitute the homogeneous polynomials

$$v_m = \frac{1}{2m+1} T_m(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; X_n, Y_n, Z_n), \tag{5.8}$$

$$w_m = \frac{3}{2m+3} T_m(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; X_n, Y_n, Z_n, P_n, P_n). \tag{5.9}$$

Recurrence relations obtained from (A.6) are valid for all m if we define $v_m = w_m = 0$, $m < 0$. Because all parameters are $\frac{1}{2}$, the coefficients in the recurrence relations are constant multiples of elementary symmetric functions E_r, and $E_1 = 0$ because $X_n + Y_n + Z_n = 0$ in (5.8) and $X_n + Y_n + Z_n + 2P_n = 0$ in (5.9). The vanishing of E_1 makes it attractive to express E_r in terms of power sums by (A.15) with $\sigma_m = 2s_n^{(m)}$. In the notation of (2.4) and (2.17) we find

$$\begin{aligned} &E_2 = -2s_n^{(2)}, \quad E_3 = 2s_n^{(3)}, \quad E_4 = 2(s_n^{(2)})^2 - 2s_n^{(4)}, \\ &E_5 = 2s_n^{(5)} - 4s_n^{(2)} s_n^{(3)}. \end{aligned} \tag{5.10}$$

Then the recurrence relations are

$$\frac{m(2m+1)}{2m-3} v_m = (2m-2) s_n^{(2)} v_{m-2} + (2m-5) s_n^{(3)} v_{m-3}, \tag{5.11}$$

$$\begin{aligned} m(2m+3) w_m = {} & (2m-1)(2m-2) s_n^{(2)} w_{m-2} + (2m-3)^2 s_n^{(3)} w_{m-3} \\ & + (2m-4)(2m-5)[s_n^{(4)} - (s_n^{(2)})^2] w_{m-4} \\ & + (2m-5)(2m-7)[s_n^{(5)} - 2s_n^{(2)} s_n^{(3)}] w_{m-5}, \end{aligned} \tag{5.12}$$

where $v_0 = w_0 = 1$ and $v_m = w_m = 0$, $m < 0$. The series in (2.5) and (2.19) come from these recurrence relations, and the asymptotic formulas for r_n in (2.6) and (2.21) are merely v_6 and w_6. The upper bounds for $|r_n|$ follow from (A.10), since

$$(\tfrac{1}{2})_6/6! = 231/1024 < \tfrac{1}{4}, \qquad (\tfrac{3}{2})_6/6! = 3003/1024 < 3.$$

The values of v_m and w_m can be checked without recurrence by using (A.12) or (A.14). For v_m it is easy to use (A.12) and (5.10) because only E_2 and E_3 are nonzero. For w_m it is more direct to put $S_m = s_n^{(m)}$ in (A.14), but $s_n^{(6)}$ must be expressed in terms of $s_n^{(2)}, \dots, s_n^{(5)}$ by using (A.15) with $E_6 = 0$ and $\sigma_m = 2s_n^{(m)}$.

Algorithm 1 reduces to Algorithm 2 if $y_0 = z_0$, which implies $y_n = z_n$, $X_n = -2Y_n = 2s_n$, $\varepsilon_n = 2|s_n|$, $s_n^{(2)} = 3s_n^2/2$, $s_n^{(3)} = s_n^3$, and

$$v_m = a_m s_n^m, \qquad a_m = \frac{1}{2m+1} T_m(\tfrac{1}{2}, 1; 2, -1). \tag{5.13}$$

To find this expression for a_m we have applied [6, Theorem 5.2–4] to (5.8). By (A.6), a_m satisfies the recurrence relation

$$\frac{m(2m+1)}{2m-1} a_m = (m-1) a_{m-1} + (2m-3) a_{m-2}, \tag{5.14}$$

where $a_0 = 1$ and $a_m = 0$, $m < 0$. By Theorem 1 in the Appendix, a_m is a dominant solution. The first iterate of (5.14) is a special case of (5.11).

The expression in parentheses on the right side of (2.10a) is

$$R_C(1, 1-t_n) = R_{-\frac{1}{2}}(\tfrac{1}{2}, 1; 1, 1-t_n) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2})_m}{m!} R_m(\tfrac{1}{2}, 1; 0, t_n). \tag{5.15}$$

The terms of this series are evaluated by [6, (6.2–5)], and the bound for $|r_n|$ in (2.11a) is obtained by noting that $1/(2m+1) \leqq 1/13$, $m \geqq 6$.

Equation (2.12) comes from [6, (6.9–16)], which implies

$$R_C(x, re^{\pm i\pi}) = \left(\frac{x}{x+r}\right)^{\frac{1}{2}} R_C(x+r, r) \mp i\frac{\pi}{2}(x+r)^{-\frac{1}{2}}, \qquad x \geqq 0,\ r > 0. \tag{5.16}$$

The Cauchy principal value of $R_C(x, -r)$ is the arithmetic mean of the values with upper and lower signs, i.e. the first term on the right side. Its asymptotic behavior for small or large positive r is easily deduced from [6, (6.9–16)]:

$$\begin{aligned} &R_C(x, -r) = \tfrac{1}{2}x^{-\frac{1}{2}} \ln(4x/r) + O(r \ln r), && x > 0,\ r \to 0, \\ &R_C(x, -r) = x^{\frac{1}{2}}/r + O(r^{-2}), && x \geqq 0,\ r \to \infty. \end{aligned} \tag{5.17}$$

From these relations and (2.22) we find

$$\begin{aligned} &R_J(x, y, z, -r) = \tfrac{3}{2}(xyz)^{-\frac{1}{2}} \ln(1/r) + O(1), && x > 0,\ r \to 0, \\ &R_J(x, y, z, -r) = -\frac{3}{r} R_F(x, y, z) + O(r^{-2}), && x \geqq 0,\ r \to \infty. \end{aligned} \tag{5.18}$$

Hence $R_J(x, y, z, -r)$ changes sign at least once on $0 < r < \infty$ if $x > 0$. Near a zero there will be cancellation on the right side of (2.22) with loss of significant figures. If $x = 0$ the sign need not change since

$$\begin{aligned} &R_C(0, -r) = 0, && r > 0, \\ &R_J(0, y, z, -r) \to -(6/yz) R_G(0, y, z) < 0, && r \to 0. \end{aligned} \tag{5.19}$$

To get the last relation we have used (2.22) and (4.14).

Algorithm 3 reduces to Algorithm 4 if $\rho_0 = z_0$, which implies $\rho_n = z_n$, $\alpha_n = \beta_n$, and $R_C(\alpha_m, \beta_m) = \beta_m^{-\frac{1}{2}}$.

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Appendix

Several new results are proved here in slightly more generality than needed for the purposes of this paper. Theorem 1 concerns a recurrence relation for the polynomial R_n generated by [6, (6.6–1)],

$$\prod_{i=1}^{k} (1-tz_i)^{-b_i} = \sum_{n=0}^{\infty} t^n \frac{(c)_n}{n!} R_n(b, z), \tag{A.1}$$

where $c = \sum_{i=1}^{k} b_i$ and $(c)_n = \Gamma(c+n)/\Gamma(c)$. It will be convenient to introduce $T_n(b, z)$, satisfying

$$\prod_{i=1}^{k} (1-tz_i)^{-b_i} = \sum_{n=-\infty}^{\infty} t^n T_n(b, z); \tag{A.2}$$

$$T_n(b, z) = \frac{(c)_n}{n!} R_n(b, z), \quad n \geqq 0; \qquad T_n = 0, \ n < 0. \tag{A.3}$$

For example $T_n(\beta, \beta; e^{i\theta}, e^{-i\theta})$, $n \geqq 0$, is the Gegenbauer polynomial $C_n^{\beta}(\cos\theta)$. The coefficients of the recurrence relation for T_n depend on the elementary symmetric functions $E_n(z)$, which are generated by a special case of (A.2):

$$\prod_{i=1}^{k} (1-tz_i) = \sum_{n=-\infty}^{\infty} t^n (-1)^n E_n(z). \tag{A.4}$$

Note that $E_n = 0$ if $n > k$ or $n < 0$. If b_i is independent of i, then T_n is symmetric in $z_1, \ldots, z_k$ and can be expressed as a polynomial in the E's; an explicit formula is given in Theorem 3. Even if the b's are not all equal, a similar formula in terms of weighted power sums is given by Theorem 4. Theorem 2 bounds the error made in truncating the series [6, (5.9–4)],

$$R_{-a}(b, 1-z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} R_n(b, z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} T_n(b, z). \tag{A.5}$$

The set of nonnegative integers will be denoted by $\mathbb{N}$, the set of all integers by $\mathbb{Z}$, the complex plane by $\mathbb{C}$, and the strictly positive real line by $\mathbb{R}_>$. If $z \in \mathbb{C}^k$ we define $1-z = (1-z_1, \ldots, 1-z_k)$ and $|z| = \max\{|z_1|, \ldots, |z_k|\}$. If $m \in \mathbb{N}^k$ we define $\|m\| = \sum_{i=1}^{k} m_i$.

It is clear from (A.2) that $T_n(b, z)$ is independent of b_j and z_j if $b_j z_j = 0$; i.e., b_j and z_j may simply be omitted if either is 0. Thus there is no loss of generality in Theorem 1 if we assume that all b's and z's are nonzero.

Theorem 1. Let $k-1 \in \mathbb{N}$ and $b, z \in (\mathbb{C} - \{0\})^k$. The polynomials $\{T_n(b, z)\colon n \in \mathbb{Z}\}$ defined by (A.2) satisfy the recurrence relations

$$\sum_{r=0}^{k} C_r T_{m-r} = 0, \quad C_r = (-1)^r \left(m - r + \sum_{i=1}^{k} b_i z_i \frac{\partial}{\partial z_i}\right) E_r, \qquad m \in \mathbb{Z}. \tag{A.6}$$

Define $c = \sum_{i=1}^{k} b_i$ and assume $-c \notin \mathbb{N}$. Then every nontrivial solution $\{y_n\colon n \in \mathbb{N}\}$ of the recurrence relations

$$\sum_{r=0}^{k} C_r y_{m-r} = 0, \qquad m \geqq k, \tag{A.7}$$

satisfies

$$\limsup_{n\to\infty} |y_n|^{1/n} = |z_i| \tag{A.8}$$

for some value of i. If $-b_j \notin \mathbb{N}$, $1 \leqq j \leqq k$, the solution $y_n = T_n$ belongs to the class of dominant solutions for which i is such that $|z_i| = |z|$.

Remarks. Although (A.6) with $m \geqq k$ is a special case of [6, (8.4–1)], the proof given here is simpler and allows $m < k$. If $\beta \in \mathbb{C} - \{0\}$ and $b_i = \beta$, $1 \leqq i \leqq k$, note that $C_r = (-1)^r (m - r + r\beta) E_r$ because E_r is homogeneous of degree r.

Proof. Let

$$g = \prod_{i=1}^{k} (1 - t z_i), \qquad G = \prod_{i=1}^{k} (1 - t z_i)^{-b_i},$$

and verify by differentiation that

$$g t \frac{\partial G}{\partial t} + G \sum_{i=1}^{k} b_i z_i \frac{\partial g}{\partial z_i} = 0.$$

By using (A.2) and (A.4) pick out the coefficient of t^m to prove (A.6). The recurrence relation is a Poincaré difference equation with characteristic polynomial

$$\sum_{r=0}^{k} (-1)^r E_r t^{k-r} = \prod_{i=1}^{k} (t - z_i).$$

Since $C_k = (-1)^k (m - k + c) E_k$, the assumption $-c \notin \mathbb{N}$ implies $C_k \neq 0$ for $m \geqq k$. Then (A.7) implies (A.8) by a theorem of Perron [13, p. 548]. If $-b_i \notin \mathbb{N}$, $1 \leqq i \leqq k$, the equation

$$\limsup_{n\to\infty} |T_n(b, z)|^{1/n} = |z|$$

is proved by observing that the reciprocal radius of convergence of the series (A.2) is the reciprocal distance from 0 to the nearest singularity of the left side.

Theorem 2. Let $a \in \mathbb{C}$ and define $\lambda = \max\{|a|, 1\}$. Let $b \in \mathbb{R}^k_>$ and $z \in \mathbb{C}^k$, and assume $|z| < 1$. Define r_n by

$$R_{-a}(b, 1 - z) = \sum_{m=0}^{n-1} \frac{(a)_m}{m!} R_m(b, z) + r_n. \tag{A.9}$$

Then

$$|r_n| \leqq \frac{(|a|)_n |z|^n}{n!(1 - |z|)^\lambda}. \tag{A.10}$$

Proof. By (A.5) and [6, (2.2–10), (6.2–24)],

$$|r_n| \leqq \sum_{m=n}^{\infty} \frac{(|a|)_m}{m!} |z|^m \leqq \frac{(|a|)_n}{n!} |z|^n \sum_{s=0}^{\infty} \frac{(|a| + n)_s}{(1 + n)_s} |z|^s. \tag{A.11}$$

If $|a| \leqq 1$ then $(|a|+n)_s/(1+n)_s \leqq 1$. If $|a| \geqq 1$ then $(|a|+n)_s/(1+n)_s \leqq (|a|)_s/(1)_s$; this is proved by multiplying the inequalities $(|a|+n+p)/(1+n+p) \leqq (|a|+p)/(1+p)$ for $p=0, 1, \ldots, s-1$. Hence the last series in (A.11) is majorized by the binomial series of $(1-|z|)^{-\lambda}$.

Theorem 3. Let $n \in \mathbb{N}$, $z \in \mathbb{C}^k$, and $\beta \in \mathbb{C}$. Define $T_n = T_n(\beta, \ldots, \beta;\ z_1, \ldots, z_k)$ by (A.2) and $E_1, \ldots, E_k$ by (A.4). Then

$$(-1)^n T_n = \sum (-1)^{\|m\|} (\beta)_{\|m\|} \frac{E_1^{m_1} \ldots E_k^{m_k}}{m_1! \ldots m_k!}, \tag{A.12}$$

where the summation extends over all $m \in \mathbb{N}^k$ such that $m_1 + 2m_2 + \ldots + k m_k = n$.

Proof. By (A.2) and (A.4) the left side is the coefficient of t^n in

$$\prod_{i=1}^{k} (1+t z_i)^{-\beta} = (1+t E_1 + \ldots + t^k E_k)^{-\beta} = \sum_{s=0}^{\infty} (-1)^s \frac{(\beta)_s}{s!} (t E_1 + \ldots + t^k E_k)^s$$

$$= \sum_{m_1=0}^{\infty} \cdots \sum_{m_k=0}^{\infty} (-1)^{\|m\|} (\beta)_{\|m\|} \frac{(t E_1)^{m_1} \ldots (t^k E_k)^{m_k}}{m_1! \ldots m_k!}.$$

In the last step we have made a multinomial expansion and changed the order of summation of the formal power series. The coefficient of t^n is the right side of (A.12).

Remarks. If $\beta = 1$, T_n is the complete symmetric function of z given by [6, (6.2–11)]. If (A.12) is divided by β, the limit as $\beta \to 0$ of the left side is the power sum $(-1)^n n^{-1} \sum_{i=1}^{k} z_i^n$ according to [6, (6.2–17)]; on the right side $\beta^{-1}(\beta)_{\|m\|} \to (\|m\| - 1)!$. Both the special case and the limiting case are well known, the latter being due to Waring in 1770.

Theorem 4. Let $n \in \mathbb{N}$ and $b, z \in \mathbb{C}^k$. Define $T_n = T_n(b, z)$ by (A.2) and weighted power sums $S_1, S_2, \ldots$ by

$$S_p = p^{-1} \sum_{i=1}^{k} b_i z_i^p, \qquad p-1 \in \mathbb{N}. \tag{A.13}$$

Then

$$T_n = \sum \frac{S_1^{m_1} \ldots S_n^{m_n}}{m_1! \ldots m_n!}, \tag{A.14}$$

where the summation extends over all $m \in \mathbb{N}^k$ such that $m_1 + 2m_2 + \ldots + n m_n = n$.

Proof. The left side is the coefficient of t^n in

$$\prod_{i=1}^{k} (1-t z_i)^{-b_i} = \exp\left[-\sum_{i=1}^{k} b_i \ln(1-t z_i)\right] = \exp\left(\sum_{p=1}^{\infty} t^p S_p\right).$$

When higher powers of t are omitted, this becomes $\exp(t S_1) \cdots \exp(t^n S_n)$. Multiply the exponential series and pick out the coefficient of t^n.

Remarks. If $b_i=-1$, $1 \leqq i \leqq k$, the left side of (A.14) is $(-1)^n E_n$. Define $\sigma_p = p^{-1} \sum_{i=1}^{k} z_i^p$, $p-1 \in \mathbb{N}$. Then

$$(-1)^n E_n = \sum (-1)^{\|m\|} \frac{\sigma_1^{m_1} \ldots \sigma_n^{m_n}}{m_1! \ldots m_n!}, \tag{A.15}$$

where the summation extends over all $m \in \mathbb{N}^k$ such that $m_1+2m_2+\ldots+nm_n=n$. If $n \leqq k$, (A.15) expresses E_n in terms of power sums (for the inverse relations see the Remarks following the proof of Theorem 3). If $n>k$ the left side is 0 and (A.15) gives σ_n in terms of $\sigma_1, \ldots, \sigma_{n-1}$. If $b_i=1$, $1 \leqq i \leqq k$, (A.14) expresses the complete symmetric function [6, (6.2–11)] in terms of power sums. Both special cases are well known.

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